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# The Cauchy problem for the integrable Novikov equation

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In this paper we consider the Cauchy problem for the integrable Novikov equation. By using the Littlewood–Paley decomposition and nonhomogeneous Besov spaces, we prove that the Cauchy problem for the integrable Novikov equation is locally well-posed in the Besov space $B^s_{p,r}$ with $1 \leqslant p, r \leqslant +\infty$ and $s > \max\{1 + \frac{1}{p}, \frac{3}{2}\}$. In particular, when $u_0 \in B^s_{p,r} \cap H^1$ with $1 \leqslant p, r \leqslant +\infty$ and $s > \max\{1 + \frac{1}{p}, \frac{3}{2}\}$, for all $t \in [0, T]$, we have that $\|u(t)\|_{H^1} = \|u_0\|_{H^1}$. We also prove that the local well-posedness of the Cauchy problem for the Novikov equation fails in $B^{3/2}_{2,\infty}$.



## 1. Introduction

Recently, Vladimir Novikov [13] found a new integrable equation:

$$u_t - u_{txx} + 4u^2 u_x - 3uu_x u_{xx} - u^2 u_{xxx} = 0. \tag{1.1}$$

It is derived that Eq. (1.1) possesses a Lax pair, many conserved densities, a Hamiltonian structure and peakon solutions $u(x,t) = \pm\sqrt{c}e^{-|x-ct-x_0|}$, $c > 0$, where $x_0$ is a constant, as well as the explicit formulas for multipeakon solutions [10,9].

The Littlewood–Paley decomposition and nonhomogeneous Besov spaces which were introduced in [16,3] have been used to establish the well-posedness of the Euler equations and the Navier–Stokes equations as well as the Camassa–Holm equation [2–7]. By using the Littlewood–Paley decomposition

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and nonhomogeneous Besov spaces as well as Kato's theory [11], Ni and Zhou [12] proved that the Cauchy problem for the Novikov equation is locally well-posed in the Besov spaces $B^{3/2}_{2,1}$ and in the Sobolev spaces $H^s(\mathbf{R})$ with $s > 3/2$ and also considered the persistence properties of the solution. Jiang and Ni [14] established some results about blow-up phenomena of the strong solution to the Cauchy problem for (1.1). Tiğlay [15] investigated the Cauchy problem for the periodic Novikov equation. Very recently, Yan, Li and Zhang [18,19] considered the Cauchy problems for the Novikov equation and weakly dissipative Novikov equation.

Since $y = u - u_{xx}$, (1.1) can be rewritten as

$$y_t + u^2 y_x + 3 y u_x u = 0, \quad t > 0.$$

We will consider the Cauchy problem for the Novikov equation:

$$y_t + u^2 y_x + 3 y u_x u = 0, \quad t > 0,\ x \in \mathbf{R}, \tag{1.2}$$

$$y = u - u_{xx}, \tag{1.3}$$

$$u(x,0) = u_0(x), \quad x \in \mathbf{R}. \tag{1.4}$$

Note that $G(x) = \frac{1}{2}e^{-|x|}$ and $G(x) * f = (1-\partial_x^2)^{-1} f$ for all $f \in L^2(\mathbf{R})$ and $G * y = u$, using (1.3), we can rewrite (1.2)–(1.4) as follows:

$$u_t + u^2 u_x + \left(1-\partial_x^2\right)^{-1}\left(3 u u_x u_{xx} + 2u_x^3 + 3u^2 u_x\right) = 0, \quad t > 0,\ x \in \mathbf{R}, \tag{1.5}$$

$$u(x,0) = u_0(x), \quad x \in \mathbf{R}. \tag{1.6}$$

The structure of the Novikov equation is complicated by comparison with the structure of equations appearing in [2,8,17]. Thus in this paper, we need to overcome some difficulties. By using the Littlewood–Paley decomposition and nonhomogeneous Besov spaces, Gui and Liu [8] considered the Cauchy problem for the two-component Camassa–Holm system. By using the Littlewood–Paley decomposition and nonhomogeneous Besov spaces, Yan and Yin [17] considered the Cauchy problem for the two-component Degasperis–Procesi system in the Besov spaces. In [8], the authors proved by induction that

$$\left\|u^n(t)\right\|_{B^s_{p,r}} + \left\|\eta^n(t)\right\|_{B^{s-1}_{p,r}} \leqslant \frac{2(\|u_0\|_{B^s_{p,r}} + \|\eta_0\|_{B^{s-1}_{p,r}})}{1 - 4C(\|u_0\|_{B^s_{p,r}} + \|\eta_0\|_{B^{s-1}_{p,r}})t} \tag{1.7}$$

with the assumption that

$$T < \min\left[\frac{1}{4C(\|u_0\|_{B^s_{p,r}} + \|\eta_0\|_{B^{s-1}_{p,r}})}, \frac{1}{2C}\right].$$

In [17], the authors proved that

$$\left\|u^n(t)\right\|_{B^s_{p,r}} + \left\|\eta^n(t)\right\|_{B^{s-1}_{p,r}} \leqslant \frac{C(\|u_0\|_{B^s_{p,r}} + \|\eta_0\|_{B^{s-1}_{p,r}})}{1 - 2C^2(\|u_0\|_{B^s_{p,r}} + \|\eta_0\|_{B^{s-1}_{p,r}})t} \tag{1.8}$$

with the assumption that

$$2C^2\left(\|u_0\|_{B^s_{p,r}} + \|\eta_0\|_{B^{s-1}_{p,r}}\right)T < 1.$$

(1.7) and (1.8) are similar to

$$\left\|u^n(t)\right\|_{B^s_{p,r}} \leqslant \frac{\|u_0\|_{B^s_{p,r}}}{1-2C\|u_0\|_{B^s_{p,r}}t} \tag{1.9}$$

which is (2.13) on page 963 in [2]. The reason is that the nonlinear terms of the equations in [2,8,17] are quadratic. In addition, in [2,8,17], the authors also used the $S^{-1}$ multiplier property of $P(D) = -\partial_x(1-\partial_x^2)^{-1}$. However, in our paper, the Novikov equation that we consider possesses cubic nonlinearity and $P(D) = -(1-\partial_x^2)^{-1}$ is an $S^{-2}$ multiplier. Moreover, the nonlinear term $(1-\partial_x^2)^{-1}(3uu_xu_{xx}+2u_x^3+3u^2u_x)$ in proving that $u^n$ is a Cauchy sequence in $B^{s-1}_{p,r}$ cannot easily be dealt with. Since the nonlinear term is cubic in (1.5), we cannot take the form similar to (1.9). In fact, we need to take

$$\left\|u^n\right\|_{B^s_{p,r}} \leqslant \frac{C\|u_0\|_{B^s_{p,r}}}{(1-4C^3\|u_0\|^2_{B^s_{p,r}}t)^{1/2}}.$$

Since $-(1-\partial_x^2)^{-1}$ is an $S^{-2}$ multiplier, we will utilize the inner relationship of the nonlinear term to prove that $u^n$ is a Cauchy sequence in $B^{s-1}_{p,r}$. More precisely, motivated by the following identity

$$\begin{aligned}
&\left(1-\partial_x^2\right)^{-1}\left(3uu_xu_{xx}+2u_x^3+3u^2u_x\right)\\
&\quad=\left(1-\partial_x^2\right)^{-1}\left[\left(\frac{3}{2}uu_x^2\right)_x+\frac{u_x^3}{2}+3u^2u_x\right]\\
&\quad=\left(1-\partial_x^2\right)^{-1}\left(\frac{3}{2}uu_x^2\right)_x+\left(1-\partial_x^2\right)^{-1}\left[\frac{u_x^3}{2}+3u^2u_x\right],
\end{aligned}$$

we obtain

$$\begin{aligned}
&\left(1-\partial_x^2\right)^{-1}\left[3u^{n+m}u_x^{n+m}u_{xx}^{n+m}+2\left(u_x^{n+m}\right)^3+3\left(u^{n+m}\right)^2u_x^{n+m}\right]\\
&\qquad-\left(1-\partial_x^2\right)^{-1}\left[3u^nu_x^nu_{xx}^n+2\left(u_x^n\right)^3+3\left(u^n\right)^2u_x^n\right]\\
&\quad=\left(1-\partial_x^2\right)^{-1}\left[3u^{n+m}u_x^{n+m}u_{xx}^{n+m}-3u^nu_x^nu_{xx}^n\right]+\left(1-\partial_x^2\right)^{-1}\left[2\left(u_x^{n+m}\right)^3-2\left(u_x^n\right)^3\right]\\
&\qquad+\left(1-\partial_x^2\right)^{-1}\left[3\left(u^{n+m}\right)^2u_x^{n+m}-3\left(u^n\right)^2u_x^n\right],
\end{aligned}\tag{1.10}$$

in (1.10), the most difficult controllable term is

$$\left(1-\partial_x^2\right)^{-1}\left[3u^{n+m}u_x^{n+m}u_{xx}^{n+m}-3u^nu_x^nu_{xx}^n\right]$$

which is

$$\begin{aligned}
&\left(1-\partial_x^2\right)^{-1}\left[3u^{n+m}u_x^{n+m}u_{xx}^{n+m}-3u^nu_x^nu_{xx}^n\right]\\
&\quad=3\left(1-\partial_x^2\right)^{-1}\left[\left(u^{n+m}-u^n\right)u_x^{n+m}u_{xx}^{n+m}+u^n\left(u_x^{n+m}-u_x^n\right)u_{xx}^{n+m}\right]\\
&\qquad+3\left(1-\partial_x^2\right)^{-1}\left[u^nu_x^n\left(u_{xx}^{n+m}-u_{xx}^n\right)\right]\\
&\quad=\frac{3}{2}\left(1-\partial_x^2\right)^{-1}\left[\left(u^{n+m}-u^n\right)\left(u_x^{n+m}\right)^2\right]_x-\frac{3}{2}\left(1-\partial_x^2\right)^{-1}\left(\left[u^{n+m}-u^n\right]_x\left(u_x^{n+m}\right)^2\right)
\end{aligned}$$

$$+3\left(1-\partial_x^2\right)^{-1}\left[u_x^{n+m}u^n\left(u^{n+m}-u^n\right)_x\right]_x-3\left(1-\partial_x^2\right)^{-1}\left[u_x^{n+m}u_x^n\left(u^{n+m}-u^n\right)_x\right]$$
$$-\frac{3}{2}\left(1-\partial_x^2\right)^{-1}\left[u^n\left[\left(u^{n+m}-u^n\right)_x\right]^2\right]_x+\frac{3}{2}\left(1-\partial_x^2\right)^{-1}\left[u_x^n\left[\left(u^{n+m}-u^n\right)_x\right]^2\right]. \tag{1.11}$$

By using (1.11), we can overcome the difficulty caused by $S^{-2}$ multiplier $(1-\partial_x^2)^{-1}$.

In this paper, motivated by [2,4], by using the Littlewood–Paley decomposition and nonhomogeneous Besov spaces, we prove that the Cauchy problem for (1.5) is locally well-posed in the Besov space $B^s_{p,r}$ with $s>\max\{1+\frac{1}{p},\frac{3}{2}\}$. In particular, when $u_0\in B^s_{p,r}\cap H^1$ with $1\leqslant p,r\leqslant+\infty$ and $s>\max\{1+\frac{1}{p},\frac{3}{2}\}$, for all $t\in[0,T]$, we have that $\|u(t)\|_{H^1}=\|u_0\|_{H^1}$. Inspired by [3], we also prove that the local well-posedness of the Cauchy problem for the Novikov equation fails in $B^{3/2}_{2,\infty}$.

To introduce the main results, we define

$$E^s_{p,r}(T)=C\left([0,T];B^s_{p,r}\right)\cap C^1\left([0,T];B^s_{p,r}\right),\quad \text{if } r<\infty,$$
$$E^s_{p,\infty}(T)=L^\infty\left(0,T;B^s_{p,\infty}\right)\cap Lip\left([0,T];B^{s-1}_{p,r}\right).$$

The main results of this paper are as follows:

**Theorem 1.1.** *Let $1\leqslant p,r\leqslant\infty$ and $s>\max(\frac{3}{2},1+\frac{1}{p})$. Let $u_0\in B^s_{p,r}$. Then there exists a time $T>0$ such that the problem* (1.5), (1.6) *has a unique solution $u$ in $E^s_{p,r}(T)$. The map $u_0\longrightarrow u$ is continuous from a neighborhood of $u_0$ in $B^s_{p,r}$ into $C([0,T];B^{s'}_{p,r})\cap C^1([0,T];B^{s'-1}_{p,r})$ for every $s'<s$. When $r<\infty$, the solution to the problem* (1.5), (1.6) *is continuous in $E^s_{p,r}(T)$.*

**Theorem 1.2.** *Let $p,r$ and $s$ be as in Theorem* 1.1. *Let $u\in E^s_{p,r}(T)$ be a solution to the problem* (1.5), (1.6) *on $[0,T]\times\mathbf{R}$ with data $u_0\in B^s_{p,r}\cap H^1$. Then the solution $u$ to the problem* (1.5), (1.6) *satisfies*

$$\forall t\in[0,T],\quad \left\|u(t)\right\|_{H^1}=\|u_0\|_{H^1}.$$

**Theorem 1.3.** *The Cauchy problem for the Novikov equation is not locally well-posed. More precisely, there exists a global solution $u\in L^\infty(\mathbf{R}^+;B^{3/2}_{2,\infty})$ to the Cauchy problem for* (1.1) *such that for any $T>0$ and $\epsilon>0$, there exists a solution $v\in L^\infty(0,T;B^{3/2}_{2,\infty})$ with*

$$\left\|v(0)-u(0)\right\|_{B^{3/2}_{2,\infty}}\leqslant\epsilon,\qquad \left\|v(t)-u(t)\right\|_{L^\infty(0,T;B^{3/2}_{2,\infty})}\geqslant 1.$$

**Remark.** Yan, Li and Zhang [19] proved that the Cauchy problem for the Novikov equation is not locally well-posed in the Sobolev spaces $H^s(\mathbf{R})$ with $s<\frac{3}{2}$ in the sense that its solutions do not depend uniformly continuously on the initial data and presented two blow-up results of strong solution to the Cauchy problem for the Novikov equation in $H^s(\mathbf{R})$ with $s>3/2$.

The remainder of this paper is organized as follows. In Section 2, we give some preliminaries. In Section 3, we establish local well-posedness of the Cauchy problem for the Novikov equation in the Besov spaces. In Section 4, we prove Theorem 1.2. In Section 5, we prove Theorem 1.3.

## 2. Preliminaries

In this section, we will recall some conclusions on the properties of the Littlewood–Paley decomposition, the nonhomogeneous Besov spaces and the theory of the transport equation which can be seen in [1–4,16].

**Lemma 2.1** *(Littlewood–Paley decomposition). There exists a couple of smooth radial functions $(\chi, \phi)$ valued in $[0, 1]$, such that $\chi$ is supported in the ball $B = \{\xi \in \mathbf{R}^n, |\xi| \leqslant \frac{4}{3}\}$ and $\phi$ is supported in the ring $C = \{\xi \in \mathbf{R}^n, \frac{3}{4} \leqslant |\xi| \leqslant \frac{8}{3}\}$. Moreover,*

$$\forall \xi \in \mathbf{R}^n, \quad \chi(\xi) + \sum_{q \in N} \phi\bigl(2^{-q}\xi\bigr) = 1$$

*and*

$$\operatorname{Supp} \phi\bigl(2^{-q}\cdot\bigr) \cap \operatorname{Supp} \phi\bigl(2^{-q'}\cdot\bigr) = \emptyset, \quad \text{if } \bigl|q - q'\bigr| \geqslant 2,$$
$$\operatorname{Supp} \chi(\cdot) \cap \operatorname{Supp} \phi\bigl(2^{-q}\cdot\bigr) = \emptyset, \quad \text{if } |q| \geqslant 1.$$

*Then for $u \in \mathscr{S}'(\mathbf{R})$, the nonhomogeneous dyadic blocks are defined as follows*:

$$\Delta_q u = 0, \quad \text{if } q \leqslant -2,$$
$$\Delta_{-1} u = \chi(D)u = \mathscr{F}_x^{-1} \chi \mathscr{F}_x u,$$
$$\Delta_q u = \phi\bigl(2^{-q}D\bigr) = \mathscr{F}_x^{-1} \phi\bigl(2^{-q}\xi\bigr) \mathscr{F}_x u, \quad \text{if } q \geqslant 0.$$

*Thus*

$$u = \sum_{q \in Z} \Delta_q u \quad \text{in } \mathscr{S}'(\mathbf{R}).$$

**Remark.** The low frequency cut-off $S_q$ is defined by

$$S_q u = \sum_{p=-1}^{q-1} \Delta_p u = \chi\bigl(2^{-q}D\bigr)u = \mathscr{F}_x^{-1} \chi\bigl(2^{-q}\xi\bigr) \mathscr{F}_x u, \quad \forall q \in N.$$

It is easily checked that

$$\Delta_p \Delta_q u \equiv 0, \quad \text{if } |p - q| \geqslant 2,$$
$$\Delta_p (S_{p-1} u \Delta_p v) \equiv 0, \quad \text{if } |p - q| \geqslant 5, \ \forall u, v \in \mathscr{S}'(\mathbf{R})$$

as well as

$$\|\Delta_p u\|_{L^p} \leqslant \|u\|_{L^p}, \qquad \|S_q u\|_{L^p} \leqslant C\|u\|_{L^p}, \quad \forall 1 \leqslant p \leqslant +\infty$$

with the aid of Young's inequality, where $C$ is a positive constant independent of $q$.

**Definition 2.1** *(Besov spaces).* Let $s \in \mathbf{R}$, $1 \leqslant p \leqslant +\infty$. The nonhomogeneous Besov space $B^s_{p,r}(\mathbf{R}^n)$ is defined by

$$B^s_{p,r}\bigl(\mathbf{R}^n\bigr) = \bigl\{ f \in \mathscr{S}'(\mathbf{R}) \colon \|f\|_{B^s_{p,r}} = \bigl\|2^{qs} \Delta_q f\bigr\|_{l^r(L^p)} = \bigl\|\bigl(2^{qs}\|\Delta_q f\|_{L^p}\bigr)_{q \geqslant -1}\bigr\|_{l^r} < \infty \bigr\}.$$

In particular, if $s = \infty$, $B^s_{p,r} = \bigcap_{s \in \mathbf{R}} B^s_{p,r}$.

**Definition 2.2.** Let $T > 0$, $s \in \mathbf{R}$ and $1 \leqslant p \leqslant \infty$. Define

$$E^s_{p,r} = \bigcap_{T>0} E^s_{p,r}(T).$$

**Lemma 2.2.** *Let $s \in \mathbf{R}$, $1 \leqslant p, r, p_j, r_j \leqslant \infty$, $j = 1, 2$, then:*

(1) *Topological properties: $B^s_{p,r}$ is a Banach space which is continuously embedded in $\mathscr{S}'(\mathbf{R})$.*
(2) *Density: $C^\infty_c$ is dense in $B^s_{p,r} \Leftrightarrow 1 \leqslant p, r < \infty$.*
(3) *Embedding: $B^s_{p_1,r_1} \hookrightarrow B^{s-n(\frac{1}{p_1}-\frac{1}{p_2})}_{p_2,r_2}$, if $p_1 \leqslant p_2$ and $r_1 \leqslant r_2$.*

$$B^{s_2}_{p,r_2} \hookrightarrow B^{s_1}_{p,r_1} \quad \textit{locally compact if } s_1 < s_2.$$

(4) *Algebraic properties: $\forall s > 0$, $B^s_{p,r} \cap L^\infty$ is an algebra. $B^s_{p,r}$ is an algebra $\Leftrightarrow B^s_{p,r} \hookrightarrow L^\infty \Leftrightarrow s > \frac{n}{p}$ or ($s \geqslant \frac{p}{n}$ and $r = 1$).*
(5) *1-D Moser-type estimates:*
   (i) *For $s > 0$,*

$$\|fg\|_{B^s_{p,r}} \leqslant C\big(\|f\|_{B^s_{p,r}}\|g\|_{L^\infty} + \|f\|_{L^\infty}\|g\|_{B^s_{p,r}}\big).$$

   (ii) *$\forall s_1 \leqslant \frac{1}{p} < s_2$ ($s_2 \geqslant \frac{1}{p}$ if $r = 1$) and $s_1 + s_2 > 0$, we have*

$$\|fg\|_{B^{s_1}_{p,r}} \leqslant C\|f\|_{B^{s_1}_{p,r}}\|g\|_{B^{s_2}_{p,r}}.$$

(6) *Complex interpolation:*

$$\|f\|_{B^{\theta s_1+(1-\theta)s_2}_{p,r}} \leqslant \|f\|^\theta_{B^{s_1}_{p,r}}\|g\|^{1-\theta}_{B^{s_2}_{p,r}}, \quad \forall f \in B^{s_1}_{p,r} \cap B^{s_2}_{p,r},\ \forall \theta \in [0,1].$$

(7) *Fatou's lemma: if $(u_n)_{n\in N}$ is bounded in $B^s_{p,r}$ and $u_n \longrightarrow u$ in $\mathscr{S}'(\mathbf{R})$, then $u \in B^s_{p,r}$ and*

$$\|u\|_{B^s_{p,r}} \leqslant \lim_{n\longrightarrow\infty} \inf \|u_n\|_{B^s_{p,r}}.$$

(8) *Let $m \in \mathbf{R}$ and $f$ be an $S^m$ multiplier (i.e., $f : \mathbf{R}^n \to \mathbf{R}$ is smooth and satisfies that $\forall \alpha \in N^n$, $\ni$ a constant $C_\alpha$, s.t. $|\partial_\alpha f(\xi)| \leqslant C_\alpha(1+|\xi|)^{m-|\alpha|}$ for all $\xi \in \mathbf{R}^n$). Then the operator $f(D)$ is continuous from $B^s_{p,r}$ to $B^{s-m}_{p,r}$.*

**Lemma 2.3** *(A prior estimates in Besov spaces). Let $1 \leqslant p, r \leqslant \infty$ and $s > -\min(\frac{1}{p}, 1-\frac{1}{p})$. Assume that $f_0 \in B^s_{p,r}$, $F \in L^1(0,T; B^s_{p,r})$ and $\partial_x v$ belongs to $L^1(0,T; B^{s-1}_{p,r})$ if $s > 1 + \frac{1}{p}$ or to $L^1(0,T; B^{1/p}_{p,r} \cap L^\infty)$ otherwise. If $f \in L^\infty(0,T; B^s_{p,r}) \cap C([0,T]; \mathscr{S}'(\mathbf{R}))$ solves the following 1-D linear transport equation:*

$$f_t + v f_x = F, \tag{2.1}$$

$$f(x,0) = f_0, \tag{2.2}$$

*then there exists a constant $C$ depending only on $s, p, r$ such that the following statements hold:*

(1) *If* $r = 1$ *or* $s \neq 1 + \frac{1}{p}$,

$$\|f\|_{B^s_{p,r}} \leqslant \|f_0\|_{B^s_{p,r}} + \int_0^t \big\|F(\tau)\big\|_{B^s_{p,r}} d\tau + C \int_0^t V'(\tau)\big\|f(\tau)\big\|_{B^s_{p,r}} d\tau$$

*or hence,*

$$\|f\|_{B^s_{p,r}} \leqslant e^{CV(t)}\left(\|f_0\|_{B^s_{p,r}} + \int_0^t e^{-CV(\tau)}\big\|F(\tau)\big\|_{B^s_{p,r}} d\tau\right) \tag{2.3}$$

*with* $V(t) = \int_0^t \|v_x(\tau)\|_{B^{1/p}_{p,r}\cap L^\infty} d\tau$ *if* $s < 1 + \frac{1}{p}$ *and* $V(t) = \int_0^t \|v_x(\tau)\|_{B^{s-1}_{p,r}} d\tau$ *else.*

(2) *If* $s \leqslant 1 + \frac{1}{p}$, $f_0' \in L^\infty$ *and* $f_x \in L^\infty((0,T)\times \mathbf{R})$ *and* $F_x \in L^1(0,T;L^\infty)$, *then*

$$\begin{aligned}&\big\|f(t)\big\|_{B^s_{p,r}} + \big\|f_x(t)\big\|_{L^\infty}\\ &\quad\leqslant e^{CV(t)}\left(\|f_0\|_{B^s_{p,r}} + \|f_{0x}\|_{L^\infty} + \int_0^t e^{-CV(\tau)}\big[\big\|F(\tau)\big\|_{B^s_{p,r}} + \big\|F_x(\tau)\big\|_{L^\infty} d\tau\big]\right)\end{aligned}$$

*with*

$$V(t) = \int_0^t \big\|\partial_x v(\tau)\big\|_{B^{1/p}_{p,r}\cap L^\infty} d\tau.$$

(3) *If* $f = v$, *then for all* $s > 0$, (1) *holds true when* $V(t) = \int_0^t \|v_x(\tau)\|_{L^\infty} d\tau$.

(4) *If* $r < \infty$, *then* $f \in C([0,T]; B^s_{p,r})$. *If* $r = \infty$, *then* $f \in C([0,T]; B^{s'}_{p,1})$ *for all* $s' < s$.

**Lemma 2.4** *(Existence and uniqueness). Let* $p, r, s, f_0$ *and* $F$ *be as in the statement of Lemma* 2.3. *Assume that* $v \in L^\rho(0,T;B^{-M}_{\infty,\infty})$ *for some* $\rho > 1$ *and* $M > 0$ *and* $v_x \in L^1(0,T;B^{s-1}_{p,r})$ *if* $s > 1 + \frac{1}{p}$ *or* $s = 1 + \frac{1}{p}$ *and* $r = 1$ *and* $v_x \in L^1(0,T;B^{1/p}_{p,\infty}\cap L^\infty)$ *if* $s < 1 + \frac{1}{p}$. *Then the problem* (2.1), (2.2) *has a unique solution* $f \in L^\infty(0,T;B^s_{p,r}) \cap (\bigcap_{s'<s} C([0,T];B^{s'}_{p,1}))$ *and the inequalities of Lemma* 2.3 *can hold true. Moreover, if* $r < \infty$, *then* $f \in C([0,T];B^s_{p,r})$.

## 3. Proof of Theorem 1.1

In this section, we define $-(1-\partial_x^2)^{-1} = P(D)$. Notice that $P(D)$ is a multiplier of degree $-2$. Now we are in a position to prove Theorem 1.1.

We will finish the proof of Theorem 1.1 with the aid of the following seven steps.

*First step: Approximate solution*

We use a standard iterative process to construct a solution. Starting from $u^0 := 0$, by induction we define a sequence of smooth functions $(u^n)_{n\in N}$ by solving the following linear transport equation:

$$\big[\partial_t + (u^n)^2\partial_x\big]u^{n+1} = P(D)\big[3u^n u^n_x u^n_{xx} + 2(u^n_x)^3 + 3(u^n)^2 u^n_x\big], \tag{3.1}$$

$$u^{n+1}(x,0) = u^{n+1}_0 = S_{n+1}u_0. \tag{3.2}$$

Since all the data belong to $B^{\infty}_{p,r}$, Lemma 2.4 enables us to show by induction that for all $n \in N$, the above equation has a global solution which belongs to $C(\mathbf{R}^+, B^{\infty}_{p,r})$.

*Second step: Uniform bounds*

We claim for all $n \in N$:

$$\left\|u^{n+1}(t)\right\|_{B^s_{p,r}} \leqslant Ce^{CU^n(t)}\left(\|u_0\|_{B^s_{p,r}} + \int_0^t e^{-CU^n(\tau)}\left\|u^n\right\|^3_{B^s_{p,r}}\,d\tau\right), \tag{3.3}$$

with $U^n = \int_0^t \|u^n\|^2_{B^s_{p,r}}\,d\tau$.

By using (2.3) of Lemma 2.3 and (3.1), we have

$$\begin{aligned}\left\|u^{n+1}(t)\right\|_{B^s_{p,r}} &\leqslant e^{C\int_0^t \|((u^n)^2)_x(t')\|_{B^{s-1}_{p,r}}\,dt'}\|u_0\|_{B^s_{p,r}} \\ &\quad + \int_0^t e^{C\int_\tau^t \|((u^n)^2)_x(t')\|_{B^{s-1}_{p,r}}\,dt'}\left\|P(D)F\left(u^n, u^n_x, u^n_{xx}\right)\right\|_{B^s_{p,r}}\,d\tau,\end{aligned} \tag{3.4}$$

where

$$F\left(u^n, u^n_x, u^n_{xx}\right) = 3u^n u^n_x u^n_{xx} + 2\left(u^n_x\right)^3 + 3\left(u^n\right)^2 u^n_x. \tag{3.5}$$

By (3.4) and the definition of the Besov spaces $B^s_{p,r}$, we have

$$\begin{aligned}\left\|u^{n+1}(t)\right\|_{B^s_{p,r}} &\leqslant Ce^{C\int_0^t \|(u^n)^2(t')\|_{B^s_{p,r}}\,dt'}\|u_0\|_{B^s_{p,r}} \\ &\quad + \int_0^t e^{C\int_t^\tau \|(u^n)^2(t')\|_{B^s_{p,r}}\,dt'}\left\|P(D)F\left(u^n, u^n_x, u^n_{xx}\right)\right\|_{B^s_{p,r}}\,d\tau.\end{aligned} \tag{3.6}$$

We also have

$$\left\|\left(u^n\right)^2\left(t'\right)\right\|_{B^s_{p,r}} \leqslant C\left\|u^n\left(t'\right)\right\|^2_{B^s_{p,r}}, \tag{3.7}$$

since $B^s_{p,r}$ is an algebra with $s > \max\{1 + \frac{1}{p}, \frac{3}{2}\}$. When $\max\{1 + \frac{1}{p}, \frac{3}{2}\} < s \leqslant 2 + \frac{1}{p}$, by using the $S^{-2}$ multiplier property of $P(D)$, the definition of the Besov spaces $B^s_{p,r}$, (ii) of (5) of Lemma 2.2 and the fact that $B^{s-1}_{p,r}$ with $s > \max\{1 + \frac{1}{p}, \frac{3}{2}\}$ is an algebra,

$$\begin{aligned}\left\|P(D)F\left(u^n, u^n_x, u^n_{xx}\right)\right\|_{B^s_{p,r}} &\leqslant C\left\|3u^n u^n_x u^n_{xx} + 2\left(u^n_x\right)^3 + 3\left(u^n\right)^2 u^n_x\right\|_{B^{s-2}_{p,r}} \\ &\leqslant C\left\|u^n_{xx}\right\|_{B^{s-2}_{p,r}}\left\|u^n u^n_x\right\|_{B^{s-1}_{p,r}} + C\left\|\left(u^n_x\right)^3\right\|_{B^{s-1}_{p,r}} + C\left\|\left(u^n\right)^3\right\|_{B^{s-1}_{p,r}} \\ &\leqslant C\left\|u^n\right\|_{B^s_{p,r}}\left\|u^n\right\|_{B^{s-1}_{p,r}}\left\|u^n_x\right\|_{B^{s-1}_{p,r}} + C\left\|u^n_x\right\|^3_{B^{s-1}_{p,r}} + C\left\|u^n\right\|^3_{B^{s-1}_{p,r}} \\ &\leqslant C\left\|u^n\right\|^3_{B^s_{p,r}}.\end{aligned} \tag{3.8}$$

When $s > 2 + \frac{1}{p}$, by using the fact that $B^{s-2}_{p,r}$ is an algebra and the standard algebra properties of the Besov spaces used in the previous paragraphs, we have

$$\left\| P(D)F\left(u^n, u^n_x, u^n_{xx}\right)\right\|_{B^s_{p,r}} \leqslant C\left\|u^n\right\|^3_{B^s_{p,r}}. \tag{3.9}$$

Inserting (3.7)–(3.9) into (3.6) yields (3.3). Thus we prove the claim.

Let us fix a $T > 0$ such that $4C^3\|u_0\|^2_{B^s_{p,r}}T < 1$ and suppose that

$$\forall t \in [0, T], \quad \left\|u^n\right\|_{B^s_{p,r}} \leqslant \frac{C\|u_0\|_{B^s_{p,r}}}{(1 - 4C^3\|u_0\|^2_{B^s_{p,r}}t)^{1/2}}. \tag{3.10}$$

Since $U^n = \int_0^t \|u^n\|^2_{B^s_{p,r}}\,d\tau$, by using (3.10), we have

$$\begin{aligned} e^{CU^n(t)-CU^n(\tau)} = e^{C\int_\tau^t \|u^n\|^2_{B^s_{p,r}}\,d\tau} &\leqslant e^{C\int_\tau^t \frac{C^2\|u_0\|^2_{B^s_{p,r}}}{(1-4C^3\|u_0\|^2_{B^s_{p,r}}t')}\,dt'} \\ &= e^{-\frac{1}{4}\int_\tau^t \frac{d(1-4C^3\|u_0\|^2_{B^s_{p,r}}t')}{1-4C^3\|u_0\|^2_{B^s_{p,r}}t'}} \\ &= e^{-\frac{1}{4}\ln\frac{1-4C^3\|u_0\|^2_{B^s_{p,r}}t}{1-4C^3\|u_0\|^2_{B^s_{p,r}}\tau}} \\ &= \left(\frac{1-4C^3\|u_0\|^2_{B^s_{p,r}}\tau}{1-4C^3\|u_0\|^2_{B^s_{p,r}}t}\right)^{1/4} \end{aligned} \tag{3.11}$$

and

$$\begin{aligned} e^{CU^n(t)} = e^{C\int_0^t \|u^n\|^2_{B^s_{p,r}}\,d\tau} &\leqslant e^{C\int_0^t \frac{C^2\|u_0\|^2_{B^s_{p,r}}}{(1-4C^3\|u_0\|^2_{B^s_{p,r}}t')}\,dt'} \\ &= e^{-\frac{1}{4}\int_0^t \frac{d(1-4C^3\|u_0\|^2_{B^s_{p,r}}t')}{1-4C^3\|u_0\|^2_{B^s_{p,r}}t'}} \\ &= e^{-\frac{1}{4}\ln(1-4C^3\|u_0\|^2_{B^s_{p,r}}t)} \\ &= \left(\frac{1}{1-4C^3\|u_0\|^2_{B^s_{p,r}}t}\right)^{1/4}. \end{aligned} \tag{3.12}$$

Inserting (3.10), (3.11) and (3.12) into (3.3) yields

$$\begin{aligned} \left\|u^{n+1}(t)\right\|_{B^s_{p,r}} &\leqslant C\left[e^{CU^n(t)}\|u_0\|_{B^s_{p,r}} + \int_0^t e^{CU^n(t)-CU^n(\tau)}\left\|u^n(\tau)\right\|^3_{B^s_{p,r}}\,d\tau\right] \\ &= C\left(\frac{1}{1-4C^3\|u_0\|^2_{B^s_{p,r}}t}\right)^{1/4}\left[\|u_0\|_{B^s_{p,r}} + \int_0^t \frac{C^3\|u_0\|^3_{B^s_{p,r}}}{(1-4C^3\|u_0\|^2_{B^s_{p,r}}\tau)^{5/4}}\,d\tau\right] \end{aligned}$$

$$\leqslant C\left(\frac{1}{1-4C^3\|u_0\|^2_{B^s_{p,r}}t}\right)^{1/4}\left[\|u_0\|_{B^s_{p,r}}-\frac{\|u_0\|_{B^s_{p,r}}}{4}\int_0^t\frac{d(1-4C^3\|u_0\|^2_{B^s_{p,r}}\tau)}{(1-4C^3\|u_0\|^2_{B^s_{p,r}}\tau)^{5/4}}\right]$$

$$=C\left(\frac{1}{1-4C^3\|u_0\|^2_{B^s_{p,r}}t}\right)^{1/2}\|u_0\|_{B^s_{p,r}}$$

$$=\frac{C\|u_0\|_{B^s_{p,r}}}{(1-4C^3\|u_0\|^2_{B^s_{p,r}}t)^{1/2}}. \tag{3.13}$$

Thus, $(u^n)_{n\in N}$ is uniformly bounded in $C([0,T];B^s_{p,r})$. By using the fact that $B^{s-1}_{p,r}$ with $s>\max\{1+\frac{1}{p},\frac{3}{2}\}$ is an algebra and $B^s_{p,r}\hookrightarrow B^{s-1}_{p,r}$ as well as the definition of the Besov spaces $B^s_{p,r}$, we have

$$\begin{aligned}\left\|\left(u^n\right)^2u_x^{n+1}\right\|_{B^{s-1}_{p,r}}&\leqslant C\left\|\left(u^n\right)^2\right\|_{B^{s-1}_{p,r}}\left\|u_x^{n+1}\right\|_{B^{s-1}_{p,r}}\\&\leqslant C\left\|u^n\right\|^2_{B^s_{p,r}}\left\|u^{n+1}\right\|_{B^s_{p,r}}\\&\leqslant\frac{C^3\|u_0\|^3_{B^s_{p,r}}}{(1-4C^3\|u_0\|^2_{B^s_{p,r}}t)^{3/2}}.\end{aligned} \tag{3.14}$$

Thus, combining (3.8), (3.9) with (3.14), we have

$$\begin{aligned}\left\|u_t^{n+1}\right\|_{B^{s-1}_{p,r}}&\leqslant\left\|\left(u^n\right)^2u_x^{n+1}\right\|_{B^{s-1}_{p,r}}+\left\|P(D)F(u,u_x,u_{xx})\right\|_{B^{s-1}_{p,r}}\\&\leqslant\left\|\left(u^n\right)^2u_x^{n+1}\right\|_{B^{s-1}_{p,r}}+\left\|P(D)F(u,u_x,u_{xx})\right\|_{B^{s}_{p,r}}\\&\leqslant(C+1)\frac{C^3\|u_0\|^3_{B^s_{p,r}}}{(1-4C^3\|u_0\|^2_{B^s_{p,r}}t)^{3/2}}.\end{aligned} \tag{3.15}$$

Consequently,

$$\left(u^n\right)_n\in C\left([0,T];B^s_{p,r}\right)\cap C^1\left([0,T];B^{s-1}_{p,r}\right). \tag{3.16}$$

*Third step: Convergence*

Now we are going to prove that $(u^n)_n$ is a Cauchy sequence in $C([0,T];B^{s-1}_{p,r})$. For $(m,n)\in N^2$, from (3.1) and (3.2), we have

$$\left(u^{n+m+1}-u^{n+1}\right)_t+\left(u^{n+m}\right)^2\left(u^{n+m+1}-u^{n+1}\right)_x=\sum_{k=1}^6T_k. \tag{3.17}$$

where

$$T_1=P(D)\left[3\left(u^{n+m}-u^n\right)u_x^{n+m}u_{xx}^{n+m}+3u^n\left(u^{n+m}-u^n\right)_xu_{xx}^{n+m}\right],$$
$$T_2=P(D)\left[3u^nu_x^n\left(u^{n+m}-u^n\right)_{xx}\right],$$
$$T_3=P(D)\left[2\left(u^{n+m}-u^n\right)_x\right]\left[\left(u_x^{n+m}\right)^2+u_x^{n+m}u_x^n+\left(u_x^n\right)^2\right],$$

$$\begin{aligned}
&T_4 = P(D)\big[3\big(u^{n+m}-u^n\big)\big(u^{n+m}+u^n\big)u_x^{n+m}\big],\\
&T_5 = P(D)\big[3\big(u^n\big)^2\big(u^{n+m}-u^n\big)_x\big],\\
&T_6 = -\big(u^{n+m}-u^n\big)\big(u^{n+m}+u^n\big)u_x^{n+1}.
\end{aligned}\tag{3.18}$$

We will estimate $\|T_j\|_{B^{s-1}_{p,r}}$ $(1\leqslant j\leqslant 6,\ j\in N)$, respectively. Since $T_j$ $(j=1,2)$ contain the second-order partial differential terms, we cannot estimate $\|T_j\|_{B^{s-1}_{p,r}}$ $(j=1,2)$, respectively. However, we notice the useful intrinsic relationship among the terms of $T_1, T_2$. Moreover, the estimation of $\|T_3\|_{B^{s-1}_{p,r}}$ is similar to the one of $\|T_5\|_{B^{s-1}_{p,r}}$ since the main difficulties in estimating $\|T_j\|_{B^{s-1}_{p,r}}$ $(j=3,5)$ that we overcome are caused by the term $(u^{n+m}-u^n)_x$.

Now we estimate $\|T_j\|_{B^{s-1}_{p,r}}$ $(1\leqslant j\leqslant 6,\ j\in N)$, respectively.

Since

$$\begin{aligned}
T_1 ={}& \frac{3}{2}P(D)\big[\big(u^{n+m}-u^n\big)\big(u_x^{n+m}\big)^2\big]_x - \frac{3}{2}P(D)\big[u^{n+m}-u^n\big]_x\big(u_x^{n+m}\big)^2\\
&+3P(D)\big[u_x^{n+m}u^n\big(u^{n+m}-u^n\big)_x\big]_x - 3P(D)\big[u_x^{n+m}u_x^n\big(u^{n+m}-u^n\big)_x\big]\\
&-3P(D)\big[u^nu_x^{n+m}\big(u^{n+m}-u^n\big)_{xx}\big],
\end{aligned}\tag{3.19}$$

thus

$$\begin{aligned}
T_1+T_2 ={}& \frac{3}{2}P(D)\big[\big(u^{n+m}-u^n\big)\big(u_x^{n+m}\big)^2\big]_x - \frac{3}{2}P(D)\big[u^{n+m}-u^n\big]_x\big(u_x^{n+m}\big)^2\\
&+3P(D)\big[u_x^{n+m}u^n\big(u^{n+m}-u^n\big)_x\big]_x - 3P(D)\big[u_x^{n+m}u_x^n\big(u^{n+m}-u^n\big)_x\big]\\
&-\frac{3}{2}P(D)\big[u^n\big[\big(u^{n+m}-u^n\big)_x\big]^2\big]_x + \frac{3}{2}P(D)\big[u_x^n\big[\big(u^{n+m}-u^n\big)_x\big]^2\big].
\end{aligned}\tag{3.20}$$

When $\max\{1+\frac{1}{p},\frac{3}{2}\}<s\leqslant 2+\frac{1}{p}$, by using the $S^{-2}$ multiplier property of $P(D)$, $B^{s-2}_{p,r}\hookrightarrow B^{s-3}_{p,r}$, (4) of Lemma 2.2, the fact that $B^{s-1}_{p,r}$ is an algebra with $s>\max\{1+\frac{1}{p},\frac{3}{2}\}$ and the definition of the Besov spaces $B^s_{p,r}$, we have

$$\begin{aligned}
\|T_1+T_2\|_{B^{s-1}_{p,r}} \leqslant{}& C\big\|\big[\big(u^{n+m}-u^n\big)\big(u_x^{n+m}\big)^2\big]_x\big\|_{B^{s-3}_{p,r}} + C\big\|\big[u^{n+m}-u^n\big]_x\big(u_x^{n+m}\big)^2\big\|_{B^{s-3}_{p,r}}\\
&+C\big\|\big[u_x^{n+m}u^n\big(u^{n+m}-u^n\big)_x\big]_x\big\|_{B^{s-3}_{p,r}} + C\big\|\big[u_x^{n+m}u_x^n\big(u^{n+m}-u^n\big)_x\big]\big\|_{B^{s-3}_{p,r}}\\
&+C\big\|\big[u^n\big[\big(u^{n+m}-u^n\big)_x\big]^2\big]_x\big\|_{B^{s-3}_{p,r}} + C\big\|u_x^n\big[\big(u^{n+m}-u^n\big)_x\big]^2\big\|_{B^{s-3}_{p,r}}\\
\leqslant{}& C\big\|\big(u^{n+m}-u^n\big)\big(u_x^{n+m}\big)^2\big\|_{B^{s-2}_{p,r}} + C\big\|\big[u^{n+m}-u^n\big]_x\big(u_x^{n+m}\big)^2\big\|_{B^{s-2}_{p,r}}\\
&+C\big\|u_x^{n+m}u^n\big(u^{n+m}-u^n\big)_x\big\|_{B^{s-2}_{p,r}} + C\big\|\big[u_x^{n+m}u_x^n\big(u^{n+m}-u^n\big)_x\big]\big\|_{B^{s-2}_{p,r}}\\
&+C\big\|\big[u^n\big[\big(u^{n+m}-u^n\big)_x\big]^2\big]\big\|_{B^{s-2}_{p,r}} + C\big\|\big[u_x^n\big[\big(u^{n+m}-u^n\big)_x\big]^2\big]\big\|_{B^{s-2}_{p,r}}\\
\leqslant{}& C\big\|\big(u^{n+m}-u^n\big)\big\|_{B^{s-2}_{p,r}}\big\|\big(u_x^{n+m}\big)^2\big\|_{B^{s-1}_{p,r}} + C\big\|\big[u^{n+m}-u^n\big]_x\big\|_{B^{s-2}_{p,r}}\big\|\big(u_x^{n+m}\big)^2\big\|_{B^{s-1}_{p,r}}\\
&+C\big\|\big(u^{n+m}-u^n\big)_x\big\|_{B^{s-2}_{p,r}}\big\|u_x^{n+m}u^n\big\|_{B^{s-1}_{p,r}} + C\big\|\big(u^{n+m}-u^n\big)_x\big\|_{B^{s-2}_{p,r}}\big\|u_x^{n+m}u_x^n\big\|_{B^{s-1}_{p,r}}\\
&+C\big\|\big[\big(u^{n+m}-u^n\big)_x\big]^2\big\|_{B^{s-2}_{p,r}}\big\|u^n\big\|_{B^{s-1}_{p,r}} + C\big\|\big[\big(u^{n+m}-u^n\big)_x\big]^2\big\|_{B^{s-2}_{p,r}}\big\|u_x^n\big\|_{B^{s-1}_{p,r}}
\end{aligned}$$

$$\begin{aligned}
&\leqslant C\|u^{n+m}-u^n\|_{B^{s-1}_{p,r}}\big[\|u^{n+m}\|^2_{B^s_{p,r}}+\|u^{n+m}\|_{B^s_{p,r}}\|u^n\|_{B^s_{p,r}}\big]\\
&\quad+C\|(u^{n+m}-u^n)_x\|_{B^{s-2}_{p,r}}\|(u^{n+m}-u^n)_x\|_{B^{s-1}_{p,r}}\|u^n\|_{B^s_{p,r}}\\
&\leqslant C\|u^{n+m}-u^n\|_{B^{s-1}_{p,r}}\big[\|u^{n+m}\|^2_{B^s_{p,r}}+\|u^n\|^2_{B^s_{p,r}}+\|u^{n+m}\|_{B^s_{p,r}}\|u^n\|_{B^s_{p,r}}\big]\\
&\leqslant C\|u^{n+m}-u^n\|_{B^{s-1}_{p,r}}\big[\|u^{n+m}\|_{B^s_{p,r}}+\|u^n\|_{B^s_{p,r}}\big]^2. \qquad (3.21)
\end{aligned}$$

When $s>2+\frac{1}{p}$, by using $B^{s-2}_{p,r}\hookrightarrow B^{s-3}_{p,r}$, the fact that $B^{s-2}_{p,r}$ is an algebra and the definition of the Besov spaces $B^s_{p,r}$ as well as $B^{s-1}_{p,r}\hookrightarrow B^{s-2}_{p,r}\hookrightarrow B^{s-3}_{p,r}$, we also can obtain the estimate of (3.21).

When $\max\{1+\frac{1}{p},\frac{3}{2}\}<s\leqslant 2+\frac{1}{p}$, by using the $S^{-2}$ multiplier property of $P(D)$, $B^{s-2}_{p,r}\hookrightarrow B^{s-3}_{p,r}$, (4) of Lemma 2.2, the fact that $B^{s-1}_{p,r}$ is an algebra with $s>\max\{1+\frac{1}{p},\frac{3}{2}\}$ and the definition of the Besov spaces $B^s_{p,r}$, we have

$$\begin{aligned}
\|T_3\|_{B^{s-1}_{p,r}}&\leqslant C\big\|\big[(u^{n+m}-u^n)_x\big]\big[(u^{n+m}_x)^2+u^{n+m}_xu^n_x+(u^n_x)^2\big]\big\|_{B^{s-3}_{p,r}}\\
&\leqslant C\big\|\big[(u^{n+m}-u^n)_x\big]\big[(u^{n+m}_x)^2+u^{n+m}_xu^n_x+(u^n_x)^2\big]\big\|_{B^{s-2}_{p,r}}\\
&\leqslant C\big\|\big[(u^{n+m}-u^n)_x\big]\big\|_{B^{s-2}_{p,r}}\big\|(u^{n+m}_x)^2+u^{n+m}_xu^n_x+(u^n_x)^2\big\|_{B^{s-1}_{p,r}}\\
&\leqslant C\|u^{n+m}-u^n\|_{B^{s-1}_{p,r}}\big[\|u^{n+m}_x\|^2_{B^{s-1}_{p,r}}+\|u^{n+m}_x\|_{B^{s-1}_{p,r}}\|u^n_x\|_{B^{s-1}_{p,r}}+\|u^n_x\|^2_{B^{s-1}_{p,r}}\big]\\
&\leqslant C\|u^{n+m}-u^n\|_{B^{s-1}_{p,r}}\big[\|u^n\|_{B^s_{p,r}}+\|u^{n+m}\|_{B^s_{p,r}}\big]^2. \qquad (3.22)
\end{aligned}$$

When $s>2+\frac{1}{p}$, by using $B^{s-2}_{p,r}\hookrightarrow B^{s-3}_{p,r}$, the fact that $B^{s-2}_{p,r}$ is an algebra and the definition of the Besov spaces $B^s_{p,r}$ as well as $B^{s-1}_{p,r}\hookrightarrow B^{s-2}_{p,r}\hookrightarrow B^{s-3}_{p,r}$, we also can obtain the estimate of (3.22).

When $\max\{1+\frac{1}{p},\frac{3}{2}\}<s\leqslant 2+\frac{1}{p}$, by using the $S^{-2}$ multiplier property of $P(D)$, $B^{s-2}_{p,r}\hookrightarrow B^{s-3}_{p,r}$, (4) of Lemma 2.2, the fact that $B^{s-1}_{p,r}$ is an algebra with $s>\max\{1+\frac{1}{p},\frac{3}{2}\}$, we have

$$\begin{aligned}
\|T_4\|_{B^{s-1}_{p,r}}&\leqslant C\big\|\big[3(u^{n+m}-u^n)(u^{n+m}+u^n)u^{n+m}_x\big]\big\|_{B^{s-3}_{p,r}}\\
&\leqslant C\big\|\big[3(u^{n+m}-u^n)(u^{n+m}+u^n)u^{n+m}_x\big]\big\|_{B^{s-2}_{p,r}}\\
&\leqslant C\big\|\big[(u^{n+m}+u^n)u^{n+m}_x\big]\big\|_{B^{s-2}_{p,r}}\big\|(u^{n+m}-u^n)\big\|_{B^{s-1}_{p,r}}\\
&\leqslant C\big\|(u^{n+m}-u^n)\big\|_{B^{s-1}_{p,r}}\|u^{n+m}_x\|_{B^{s-2}_{p,r}}\big\|(u^{n+m}+u^n)\big\|_{B^{s-1}_{p,r}}\\
&\leqslant C\big\|(u^{n+m}-u^n)\big\|_{B^{s-1}_{p,r}}\|u^{n+m}\|_{B^{s-1}_{p,r}}\big(\|u^{n+m}\|_{B^{s-1}_{p,r}}+\|u^n\|_{B^{s-1}_{p,r}}\big)\\
&\leqslant C\big\|(u^{n+m}-u^n)\big\|_{B^{s-1}_{p,r}}\|u^{n+m}\|_{B^s_{p,r}}\big(\|u^{n+m}\|_{B^s_{p,r}}+\|u^n\|_{B^s_{p,r}}\big)\\
&\leqslant C\|u^{n+m}-u^n\|_{B^{s-1}_{p,r}}\big[\|u^n\|_{B^s_{p,r}}+\|u^{n+m}\|_{B^s_{p,r}}\big]^2. \qquad (3.23)
\end{aligned}$$

When $s>2+\frac{1}{p}$, by using $B^{s-2}_{p,r}\hookrightarrow B^{s-3}_{p,r}$, the fact that $B^{s-2}_{p,r}$ is an algebra and the definition of the Besov spaces $B^s_{p,r}$ as well as $B^{s-1}_{p,r}\hookrightarrow B^{s-2}_{p,r}\hookrightarrow B^{s-3}_{p,r}$, we also can obtain the estimate of (3.23).

By using the estimation similar to $\|T_3\|_{B^{s-1}_{p,r}}$, we can obtain

$$\|T_5\|_{B^{s-1}_{p,r}}\leqslant C\|u^{n+m}-u^n\|_{B^{s-1}_{p,r}}\big(\|u^{n+m}\|_{B^s_{p,r}}+\|u^n\|_{B^s_{p,r}}\big)^2. \qquad (3.24)$$

By using the fact that $B^{s-1}_{p,r}$ with $s > 1 + \frac{1}{P}$ is a Banach algebra, we can obtain

$$\|T_6\|_{B^{s-1}_{p,r}} \leqslant C\big\|u^{n+m} - u^n\big\|_{B^{s-1}_{p,r}}\big(\big\|u^{n+m}\big\|_{B^s_{p,r}} + \big\|u^n\big\|_{B^s_{p,r}}\big)^2. \tag{3.25}$$

Gathering (3.21)–(3.25) together, we have

$$\begin{aligned}\left\|\sum_{j=1}^{6} T_j\right\|_{B^{s-1}_{p,r}} &\leqslant \sum_{j=1}^{6}\|T_j\|_{B^{s-1}_{p,r}} \\ &\leqslant C\big\|u^{n+m} - u^n\big\|_{B^{s-1}_{p,r}}\big[\big\|u^n\big\|_{B^s_{p,r}} + \big\|u^{n+m}\big\|_{B^s_{p,r}}\big]^2 \\ &\quad + C\big\|u^{n+m} - u^n\big\|_{B^{s-1}_{p,r}}\big\|u^{n+1}\big\|_{B^s_{p,r}}\big[\big\|u^n\big\|_{B^s_{p,r}} + \big\|u^{n+m}\big\|_{B^s_{p,r}}\big].\end{aligned} \tag{3.26}$$

When $s \neq 2 + \frac{1}{p}$, combining Lemma 2.4 with (3.17), we have

$$\begin{aligned}\big\|u^{n+m+1}(t) - u^{n+1}(t)\big\|_{B^{s-1}_{p,r}} &\leqslant Ce^{CU^{n+m}(t)}\big\|\big(u^{n+m+1} - u^{n+1}\big)(\cdot,0)\big\|_{B^{s-1}_{p,r}} \\ &\quad + C\int_0^t e^{CU^{n+m}(t)-CU^{n+m}(\tau)}\sum_{j=1}^{6}\|T_j\|_{B^{s-1}_{p,r}}\,d\tau\end{aligned} \tag{3.27}$$

where

$$U^{n+m}(t) = \int_0^t \big\|\partial_x\big(u^{n+m}\big)^2(\tau)\big\|_{B^{\frac{1}{p}}_{p,r}\cap L^\infty}\,d\tau \tag{3.28}$$

if $s - 1 < 1 + \frac{1}{p}$ and

$$U^{n+m}(t) = \int_0^t \big\|\partial_x\big(u^{n+m}\big)^2(\tau)\big\|_{B^{s-2}_{p,r}}\,d\tau \tag{3.29}$$

if $s - 1 > 1 + \frac{1}{p}$. From (3.28), if $s - 1 < 1 + \frac{1}{p}$, by using $B^{s-1}_{p,r} \hookrightarrow L^\infty$ with $s > 1 + \frac{1}{p}$, we have

$$U^{n+m}(t) \leqslant C\int_0^t \big\|u^{n+m}(\tau)\big\|^2_{B^s_{p,r}}\,d\tau. \tag{3.30}$$

From (3.29), if $s - 1 > 1 + \frac{1}{p}$, we have

$$U^{n+m}(t) \leqslant C\int_0^t \big\|u^{n+m}(\tau)\big\|^2_{B^s_{p,r}}\,d\tau. \tag{3.31}$$

Since

$$\Big\|\sum_{q=n+1}^{n+m}\Delta_q u_0\Big\|_{B^{s-1}_{p,r}} \leqslant C2^{-n}\|u_0\|_{B^{s-1}_{p,r}} \tag{3.32}$$

which can be seen on page 2142 of [17], and $(u^n)_n$ is uniformly bounded in $E^S_{p,r}(T)$, by using (3.26)–(3.32), we have

$$\big\|u^{n+m+1}(t)-u^{n+1}(t)\big\|_{B^{s-1}_{p,r}} \leqslant C_T\left(2^{-n}+\int_0^t \big\|u^{n+m+1}(\tau)-u^{n+1}(\tau)\big\|_{B^{s-1}_{p,r}}\,d\tau\right). \tag{3.33}$$

It is easily checked by induction

$$\big\|u^{n+m+1}-u^{n+1}\big\|_{L^\infty_T B^{s-1}_{p,r}} \leqslant \frac{(TC_T)^{n+1}}{(n+1)!}\big\|u^m\big\|_{B^{s-1}_{p,r}}+2^{-n}C_T\sum_{k=0}^{n}2^k\frac{(TC_T)^k}{k!}. \tag{3.34}$$

Since $\|u^m\|_{L^\infty_T(B^{s-1}_{p,r})}$ is bounded independently of $m$, we can find a new constant $C'_T$ such that

$$\big\|u^{n+m+1}-u^{n+1}\big\|_{L^\infty_T B^{s-1}_{p,r}} \leqslant C'_T 2^{-n}. \tag{3.35}$$

Consequently, $(u^n)_n$ is a Cauchy sequence in $C([0,T];B^{s-1}_{p,r})$, moreover, $(u^n)_n$ converges to some limit function $u\in C([0,T];B^{s-1}_{p,r})$.

When $s=2+\frac{1}{p}$, by using (6) of Lemma 2.2, we have

$$\begin{aligned}
\big\|u^{n+m+1}-u^{n+1}\big\|_{L^\infty_T B^{s-1}_{p,r}} &= \big\|u^{n+m+1}-u^{n+1}\big\|_{L^\infty_T B^{1+\frac{1}{p}}_{p,r}}\\
&\leqslant \big\|u^{n+m+1}-u^{n+1}\big\|^{\theta}_{L^\infty_T B^{s_1}_{p,r}}\big\|u^{n+m+1}-u^{n+1}\big\|^{1-\theta}_{L^\infty_T B^{s_2}_{p,r}}\\
&\leqslant \big\|u^{n+m+1}(t)-u^{n+1}(t)\big\|^{\theta}_{B^{1+\frac{1}{p}}_{p,r}}\Big[\big\|u^{n+m+1}\big\|_{B^{2+\frac{1}{p}}_{p,r}}+\big\|u^{n+1}\big\|_{B^{2+\frac{1}{p}}_{p,r}}\Big]^{1-\theta}\\
&\leqslant \big(C'_T\big)^{\theta}2^{-\theta n}\Big[\big\|u^{n+m+1}\big\|_{B^{2+\frac{1}{p}}_{p,r}}+\big\|u^{n+1}(t)\big\|_{B^{2+\frac{1}{p}}_{p,r}}\Big]^{1-\theta}
\end{aligned} \tag{3.36}$$

where

$$s_1\in\left(\max\left(1+\frac{1}{p},\frac{3}{2}\right)-1,1+\frac{1}{p}\right),\qquad s_2\in\left(1+\frac{1}{p},2+\frac{1}{p}\right).$$

Consequently, $(u^n)_n$ is a Cauchy sequence in $C([0,T];B^{s-1}_{p,r})$, moreover, $(u^n)_n$ converges to some limit function $u\in C([0,T];B^{s-1}_{p,r})$.

*Fourth step: Existence of solution in $E^s_{p,r}(T)$*

We prove that $u$ belongs to $E_{p,r}(T)$ and satisfies the Novikov equation (1.5), (1.6). Since $(u^n)_n$ is uniformly bounded in $L^\infty(0,T;B^s_{p,r})$, Fatou's property for the Besov spaces guarantees that $u \in L^\infty(0,T;B^s_{p,r})$. If $s' \leqslant s-1$, then

$$\|u^n - u\|_{B^{s'}_{p,r}} \leqslant \|u^n - u\|_{B^{s-1}_{p,r}}. \tag{3.37}$$

If $s-1 < s' < s$, by using (6) of Lemma 2.2, we have

$$\begin{aligned} \|u^n - u\|_{B^{s'}_{p,r}} &\leqslant \|u^n - u\|^{\theta'}_{B^{s-1}_{p,r}} \|u^n - u\|^{1-\theta}_{B^s_{p,r}} \\ &\leqslant \|u^n - u\|^{\theta'}_{B^{s-1}_{p,r}} \big(\|u\|_{B^s_{p,r}} + \|u^n\|_{B^s_{p,r}}\big)^{1-\theta}, \end{aligned} \tag{3.38}$$

where

$$\theta' = s - s'.$$

Combining (3.37) with (3.38), for all $s' < s$, we have that $(u^n)_n$ converges to $u$ in $C([0,T];B^{s'}_{p,r})$. Taking limit in (3.1) and (3.2), we conclude that $u$ is indeed a solution of (1.5), (1.6). Now $u \in L^\infty(0,T;B^s_{p,r})$, the right-hand side of the following equation

$$u_t + u^2 u_x = P(D)\big(3u_x u u_{xx} + 2u_x^3 + 3u^2 u_x\big), \tag{3.39}$$

also belongs to $L^\infty(0,T;B^s_{p,r})$ since

$$\begin{aligned} \big\|P(D)\big(3u_x u u_{xx} + 2u_x^3 + 3u^2 u_x\big)\big\|_{B^s_{p,r}} &= \left\|P(D)\left(\frac{3}{2}\big(uu_x^2\big)_x + \frac{1}{2}u_x^3 + \big(u^3\big)_x\right)\right\|_{B^s_{p,r}} \\ &\leqslant C\left\|\frac{3}{2}\big(uu_x^2\big)_x + \frac{1}{2}u_x^3 + \big(u^3\big)_x\right\|_{B^{s-2}_{p,r}} \\ &\leqslant C\|uu_x^2\|_{B^{s-1}_{p,r}} + C\|(u_x)^3\|_{B^{s-1}_{p,r}} + C\|u^3\|_{B^{s-1}_{p,r}} \\ &\leqslant C\|u\|_{B^{s-1}_{p,r}}\|u_x^2\|_{B^{s-1}_{p,r}} + C\|(u_x)\|^3_{B^{s-1}_{p,r}} + C\|u\|^3_{B^{s-1}_{p,r}} \\ &\leqslant C\|u\|^3_{B^s_{p,r}}, \end{aligned} \tag{3.40}$$

in the above process of calculation, we have used the $S^{-2}$ multiplier property of $P(D)$, $B^{s-1}_{p,r} \hookrightarrow B^{s-2}_{p,r}$, the definition of the Besov spaces $B^s_{p,r}$ and the fact that $B^{s-1}_{p,r}$ is an algebra with $s > \max\{1+\frac{1}{p}, \frac{3}{2}\}$. When $r < \infty$, from Lemma 2.4, we know that $u \in C([0,T];B^s_{p,r})$. By using (3.39), we have that $u_t \in C([0,T];B^{s-1}_{p,r})$ and in $L^\infty([0,T];B^{s-1}_{p,r})$ otherwise. Consequently, $u \in E^s_{p,r}(T)$.

*Fifth step: Uniqueness of solution*

We firstly consider the case $s \neq 2 + \frac{1}{p}$. Suppose that $u, v \in L^\infty(0,T;B^s_{p,r}) \cap C([0,T];B^{s-1}_{p,r})$ are the solution to the Novikov equation with initial data $u_0, v_0 \in B^s_{p,r}$, respectively. We claim that

$$\|u(t) - v(t)\|_{B^{s-1}_{p,r}} \leqslant \|u_0 - v_0\|_{B^{s-1}_{p,r}} e^{C\int_0^t (\|u\|_{B^s_{p,r}} + \|v\|_{B^s_{p,r}})^2(\tau)\,d\tau}. \tag{3.41}$$

It is easily checked that $w = u - v$ satisfies the following transport equation

$$w_t + v^2 w_x = t_1 + t_2 + t_3 - (u+v)u_x w \tag{3.42}$$

where

$$\begin{aligned}
t_1 &= P(D)\left[\left(\frac{3}{2}wu_x^2 + 3vu_xw_x - \frac{3}{2}vw_x^2\right)_x - \frac{3}{2}u_x^2 w_x - 3u_xv_xw_x + \frac{3}{2}v_xw_x^2\right],\\
t_2 &= P(D)\left[2w_x\left(u_x^2 + u_xv_x + v_x^2\right)\right],\\
t_3 &= P(D)\left[3(u+v)u_xw + 3v^2w_x\right].
\end{aligned}$$

When $s - 1 < 1 + \frac{1}{p}$, applying Lemma 2.3 to (3.42) leads to

$$\begin{aligned}
\left\|w(t)\right\|_{B^{s-1}_{p,r}} &\leqslant \|w_0\|_{B^{s-1}_{p,r}} e^{C\int_0^t \|(v^2)_x(\tau')\|_{B^{\frac{1}{p}}_{p,r}\cap L^\infty} d\tau'} + \int_0^t e^{C\int_\tau^t \|(v^2)_x(\tau')\|_{B^{\frac{1}{p}}_{p,r}\cap L^\infty} d\tau'}\\
&\quad \times \left(\left\|(u+v)u_xw\right\|_{B^{s-1}_{p,r}} + \|t_1+t_2+t_3\|_{B^{s-1}_{p,r}}\right).
\end{aligned} \tag{3.43}$$

From (3.43), if $s - 1 < 1 + \frac{1}{p}$, by using $B^{s-1}_{p,r} \hookrightarrow L^\infty$ with $s - 1 > \frac{1}{p}$, we have

$$\begin{aligned}
\left\|w(t)\right\|_{B^{s-1}_{p,r}} &\leqslant \|w_0\|_{B^{s-1}_{p,r}} e^{C\int_0^t \|v(\tau')\|^2_{B^s_{p,r}} d\tau'} + \int_0^t e^{C\int_\tau^t \|v(\tau')\|^2_{B^s_{p,r}} d\tau'}\\
&\quad \times \left(\left\|(u+v)u_xw\right\|_{B^{s-1}_{p,r}} + \|t_1+t_2+t_3\|_{B^{s-1}_{p,r}}\right).
\end{aligned} \tag{3.44}$$

When $s - 1 > 1 + \frac{1}{p}$, applying Lemma 2.3 to (3.42) leads to

$$\begin{aligned}
\left\|w(t)\right\|_{B^{s-1}_{p,r}} &\leqslant \|w_0\|_{B^{s-1}_{p,r}} e^{C\int_0^t \|(v^2)_x(\tau')\|_{B^{s-2}_{p,r}} d\tau'} + C\int_0^t e^{C\int_\tau^t \|(v^2)_x(\tau')\|_{B^{s-2}_{p,r}} d\tau'}\\
&\quad \times \left(\left\|(u+v)u_xw\right\|_{B^{s-1}_{p,r}} + \|t_1+t_2+t_3\|_{B^{s-1}_{p,r}}\right).
\end{aligned} \tag{3.45}$$

From (3.45), if $s - 1 > 1 + \frac{1}{p}$, by using $B^{s-1}_{p,r} \hookrightarrow L^\infty$ with $s - 1 > \frac{1}{p}$, we have

$$\begin{aligned}
\left\|w(t)\right\|_{B^{s-1}_{p,r}} &\leqslant C\|w_0\|_{B^{s-1}_{p,r}} e^{C\int_0^t \|v(\tau')\|^2_{B^s_{p,r}} d\tau'} + C\int_0^t e^{C\int_\tau^t \|v(\tau')\|^2_{B^s_{p,r}} d\tau'}\\
&\quad \times \left(\left\|(u+v)u_xw\right\|_{B^{s-1}_{p,r}} + \|t_1+t_2+t_3\|_{B^{s-1}_{p,r}}\right).
\end{aligned} \tag{3.46}$$

By using the definition of the Besov spaces $B^s_{p,r}$, $S^{-2}$ multiplier property of $P(D)$ and $s > \max\{1 + \frac{1}{p}, \frac{3}{2}\}$ which leads to the fact that $B^{s-1}_{p,r}$ is an algebra, we have

$$\left\|\left(v^2\right)_x\left(\tau'\right)\right\|_{B^{s-1}_{p,r}} \leqslant C\left\|v^2\right\|_{B^s_{p,r}} \leqslant C\|v\|^2_{B^s_{p,r}}, \tag{3.47}$$

$$\begin{aligned}\left\|(u+v)u_x w\right\|_{B^{s-1}_{p,r}} &\leqslant C\|u+v\|_{B^{s-1}_{p,r}}\|u_x\|_{B^{s-1}_{p,r}}\|w\|_{B^{s-1}_{p,r}}\\ &\leqslant C\|w\|_{B^{s-1}_{p,r}}\left[\|u\|^2_{B^s_{p,r}}+\|u\|_{B^s_{p,r}}\|v\|_{B^s_{p,r}}\right].\end{aligned} \tag{3.48}$$

When $\max\{1+\frac{1}{p},\frac{3}{2}\}<s\leqslant 2+\frac{1}{p}$, by using the $S^{-2}$ multiplier property of $P(D)$, $B^{s-2}_{p,r}\hookrightarrow B^{s-3}_{p,r}$, (4) of Lemma 2.2, the fact that $B^{s-1}_{p,r}$ is an algebra with $s>\max\{1+\frac{1}{p},\frac{3}{2}\}$ and the definition of the Besov spaces $B^s_{p,r}$, we have

$$\begin{aligned}\|t_1\|_{B^{s-1}_{p,r}} &\leqslant \left\|\frac{3}{2}wu_x^2+3vu_xw_x-\frac{3}{2}vw_x^2\right\|_{B^{s-2}_{p,r}}+\left\|-\frac{3}{2}u_x^2w_x-3u_xv_xw_x+\frac{3}{2}v_xw_x^2\right\|_{B^{s-2}_{p,r}}\\ &\leqslant C\left\|wu_x^2\right\|_{B^{s-2}_{p,r}}+C\|vu_xw_x\|_{B^{s-2}_{p,r}}+C\left\|vw_x^2\right\|_{B^s_{p,r}}\\ &\quad+C\left\|u_x^2w_x\right\|_{B^{s-2}_{p,r}}+C\|u_xv_xw_x\|_{B^{s-2}_{p,r}}+C\left\|v_xw_x^2\right\|_{B^{s-2}_{p,r}}\\ &\leqslant C\|w\|_{B^{s-2}_{p,r}}\left\|(u_x)^2\right\|_{B^{s-1}_{p,r}}+C\|w_x\|_{B^{s-2}_{p,r}}\|u_xv\|_{B^{s-1}_{p,r}}+C\|w_x\|_{B^{s-2}_{p,r}}\|vw_x\|_{B^{s-1}_{p,r}}\\ &\quad+C\|w_x\|_{B^{s-2}_{p,r}}\|u_x\|^2_{B^{s-1}_{p,r}}+C\|w_x\|_{B^{s-2}_{p,r}}\|u_x\|_{B^{s-1}_{p,r}}\|v_x\|_{B^{s-1}_{p,r}}+C\|w_x\|_{B^{s-2}_{p,r}}\|v_xw_x\|_{B^{s-2}_{p,r}}\\ &\leqslant C\|w\|_{B^{s-1}_{p,r}}\left(\|u\|_{B^s_{p,r}}+\|v\|_{B^s_{p,r}}\right)^2.\end{aligned} \tag{3.49}$$

When $s>2+\frac{1}{p}$, by using $B^{s-2}_{p,r}\hookrightarrow B^{s-3}_{p,r}$, the fact that $B^{s-2}_{p,r}$ is an algebra and the definition of the Besov spaces $B^s_{p,r}$ as well as $B^{s-1}_{p,r}\hookrightarrow B^{s-2}_{p,r}\hookrightarrow B^{s-3}_{p,r}$, we also can obtain the estimate of (3.49).

When $\max\{1+\frac{1}{p},\frac{3}{2}\}<s\leqslant 2+\frac{1}{p}$, by using the $S^{-2}$ multiplier property of $P(D)$, $B^{s-2}_{p,r}\hookrightarrow B^{s-3}_{p,r}$, (4) of Lemma 2.2, the fact that $B^{s-1}_{p,r}$ is an algebra with $s>\max\{1+\frac{1}{p},\frac{3}{2}\}$ and the definition of the Besov spaces $B^s_{p,r}$, we have

$$\begin{aligned}\|t_2\|_{B^{s-1}_{p,r}} &\leqslant C\left\|w_x\left(u_x^2+u_xv_x+v_x^2\right)\right\|_{B^{s-2}_{p,r}}\\ &\leqslant C\|w_x\|_{B^{s-2}_{p,r}}\left\|u_x^2+u_xv_x+v_x^2\right\|_{B^{s-1}_{p,r}}\\ &\leqslant C\|w\|_{B^{s-1}_{p,r}}\left(\|u\|_{B^s_{p,r}}+\|v\|_{B^s_{p,r}}\right)^2.\end{aligned} \tag{3.50}$$

When $s>2+\frac{1}{p}$, by using $B^{s-2}_{p,r}\hookrightarrow B^{s-3}_{p,r}$, the fact that $B^{s-2}_{p,r}$ is an algebra and the definition of the Besov spaces $B^s_{p,r}$ as well as $B^{s-1}_{p,r}\hookrightarrow B^{s-2}_{p,r}\hookrightarrow B^{s-3}_{p,r}$, we also can obtain the estimate of (3.50).

When $\max\{1+\frac{1}{p},\frac{3}{2}\}<s\leqslant 2+\frac{1}{p}$, by using the $S^{-2}$ multiplier property of $P(D)$, $B^{s-2}_{p,r}\hookrightarrow B^{s-3}_{p,r}$, (4) of Lemma 2.2, the fact that $B^{s-1}_{p,r}$ is an algebra with $s>\max\{1+\frac{1}{p},\frac{3}{2}\}$ and the definition of the Besov spaces $B^s_{p,r}$, we have

$$\begin{aligned}\|t_3\|_{B^{s-2}_{p,r}} &\leqslant \left\|(u+v)u_xw+v^2w_x\right\|_{B^{s-2}_{p,r}}\\ &\leqslant C\|w\|_{B^{s-2}_{p,r}}\left\|(u+v)u_x\right\|_{B^{s-1}_{p,r}}+C\|w_x\|_{B^{s-2}_{p,r}}\left\|v^2\right\|_{B^{s-1}_{p,r}}\\ &\leqslant C\|w\|_{B^{s-1}_{p,r}}\left(\|u\|_{B^s_{p,r}}+\|v\|_{B^s_{p,r}}\right)^2.\end{aligned} \tag{3.51}$$

When $s > 2 + \frac{1}{p}$, by using $B^{s-2}_{p,r} \hookrightarrow B^{s-3}_{p,r}$, the fact that $B^{s-2}_{p,r}$ is an algebra and the definition of the Besov spaces $B^{s}_{p,r}$ as well as $B^{s-1}_{p,r} \hookrightarrow B^{s-2}_{p,r} \hookrightarrow B^{s-3}_{p,r}$, we also can obtain the estimate of (3.51). Inserting (3.47)–(3.51) into (3.43) and applying Gronwall's inequality yields (3.41). In particular, $u_0 = v_0$ in (3.41) yields $u(t) = v(t)$.

For $s = 2 + \frac{1}{p}$, combining the interpolation with the case $s \neq 2 + \frac{1}{p}$, we can easily obtain the uniqueness result.

**Remark.** If $v_0$ is in a small neighborhood of $u_0$, (3.41) gives the existence of a solution $v \in E^{s}_{p,r}(T)$.

*Sixth step: Continuity with respect to the initial data*

When $s' = s - 1$, the conclusion is valid. When $s' < s - 1$, by using (6) of Lemma 2.2 and (3.41), we have

$$\begin{aligned}
\|u(t) - v(t)\|_{B^{s'}_{p,r}} &\leqslant C\|u(t) - v(t)\|^{\theta_1}_{B^{s_1}_{p,r}} \|u(t) - v(t)\|^{1-\theta_1}_{B^{s-1}_{p,r}} \\
&\leqslant C\|u(t) - v(t)\|^{\theta_1}_{B^{s_1}_{p,r}} e^{C(1-\theta_1)\int_0^t (\|u\|_{B^s_{p,r}} + \|v\|_{B^s_{p,r}})^2(\tau)\,d\tau} \|u_0 - v_0\|^{1-\theta_1}_{B^{s-1}_{p,r}} \\
&\leqslant C\|u(t) - v(t)\|^{\theta_1}_{B^{s_1}_{p,r}} e^{C(1-\theta_1)\int_0^t (\|u\|_{B^s_{p,r}} + \|v\|_{B^s_{p,r}})^2(\tau)\,d\tau} \\
&\quad \times \|u_0 - v_0\|^{(1-\theta_1)\theta_2}_{B^{s'}_{p,r}} \|u_0 - v_0\|^{(1-\theta_1)(1-\theta_2)}_{B^{s}_{p,r}} \\
&\leqslant C\big[\|u(t)\|_{B^s_{p,r}} + \|v(t)\|_{B^s_{p,r}}\big]^{\theta_1} e^{C(1-\theta_1)\int_0^t (\|u\|_{B^s_{p,r}} + \|v\|_{B^s_{p,r}})^2(\tau)\,d\tau} \\
&\quad \times \|u_0 - v_0\|^{(1-\theta_1)\theta_2}_{B^{s'}_{p,r}} \big[\|u_0\|_{B^s_{p,r}} + \|v_0\|_{B^s_{p,r}}\big]^{(1-\theta_1)(1-\theta_2)}
\end{aligned} \tag{3.52}$$

where $s' = \theta_1 s_1 + (1 - \theta_1)(s - 1)$, $s_1 < s - 1$, $s - 1 = \theta_2 s' + s(1 - \theta_2)$.

When $s - 1 < s' < s$, by using (6) of Lemma 2.2 and (3.41), we have

$$\begin{aligned}
\|u(t) - v(t)\|_{B^{s'}_{p,r}} &\leqslant C\|u(t) - v(t)\|^{\theta_3}_{B^{s-1}_{p,r}} \|u(t) - v(t)\|^{1-\theta_3}_{B^{s}_{p,r}} \\
&\leqslant C\|u(0) - v(0)\|^{\theta_3}_{B^{s-1}_{p,r}} e^{C\theta_3\int_0^t (\|u\|_{B^s_{p,r}} + \|v\|_{B^s_{p,r}})^2(\tau)\,d\tau} \big[\|u(t)\|_{B^s_{p,r}} + \|v(t)\|_{B^s_{p,r}}\big]^{1-\theta_3} \\
&\leqslant C\|u(0) - v(0)\|^{\theta_3}_{B^{s'}_{p,r}} e^{C\theta_3\int_0^t (\|u\|_{B^s_{p,r}} + \|v\|_{B^s_{p,r}})^2(\tau)\,d\tau} \big[\|u(t)\|_{B^s_{p,r}} + \|v(t)\|_{B^s_{p,r}}\big]^{1-\theta_3}
\end{aligned} \tag{3.53}$$

where $s' = \theta_3(s - 1) + (1 - \theta_3)s$.

*Seventh step: Continuity*

The continuity in $C([0, T]; B^s_{p,r}) \cap C([0, T]; B^{s-1}_{p,r})$ if $r < \infty$ is valid since we can finish the proof of continuity by using the sequence of viscous approximate solutions $(u_\epsilon)_{\epsilon>0}$ for the Novikov equation which converges in $C([0, T]; B^s_{p,r}) \cap C([0, T]; B^{s-1}_{p,r})$.

Consequently, we complete the proof of Theorem 1.1.

## 4. Proof of Theorem 1.2

In this section, we will prove Theorem 1.2.

More precisely, in this section, we will prove that when $u_0 \in H^1 \cap B^s_{p,r}$ with $s > \max\{1 + \frac{1}{p}, \frac{3}{2}\}$, and $u$ is defined on $[0, T]$ and $u \in E^s_{p,r}(T)$, $\|u(t)\|_{H^1} = \|u_0\|_{H^1}$. Now we define $u^n_0 = \rho^n * u_0$, where $(\rho^n)_n$ is a sequence of nonnegative mollifiers. It is easily checked that

$$\left\|u^n_0\right\|_{B^s_{p,r}} \leqslant \|u_0\|_{B^s_{p,r}}, \tag{4.1}$$

$$\left\|u^n_0\right\|_{H^1} \leqslant \|u_0\|_{H^1}. \tag{4.2}$$

Since $u^n_0 \in H^4$, from Theorem 1.1, we have that there exists $T$ such that $4C^3 T\|u_0\|^2_{B^s_{p,r}} \leqslant 1$ and $u^n \in C([0, T]; H^4 \cap B^s_{p,r}) \cap C([0, T]; H^3 \cap B^{s-1}_{p,r})$. In particular, from (4.2), we have

$$\left\|u^n(t)\right\|_{H^1} = \left\|u^n_0\right\|_{H^1} \leqslant \|u_0\|_{H^1}. \tag{4.3}$$

Applying Lemma 2.3 to the Novikov equation satisfying by $u^n$ and following the proof of Theorem 1.1, we know that $(u^n)_n$ converges to $u$ which is the solution of the Novikov equation. Combining (4.3) with Fatou's property, we have

$$\left\|u(t)\right\|_{H^1} \leqslant \|u_0\|_{H^1}. \tag{4.4}$$

Since $-u(-x, -t)$ is the solution of the Novikov equation, we can solve the Novikov equation backward, starting from $u(T')$ for $T' < T$, this yields a solution $v \in E^s_{p,r}(T')$ defined on $[0, T']$. By using the uniqueness since $s > \frac{3}{2}$, we have that $v \equiv u$ on $[0, T']$, hence $\|v(0)\|_{H^1} \leqslant \|u(T')\|_{H^1}$. Thus for $t \in [0, T']$, we have $\|u(0)\|_{H^1} = \|u(t)\|_{H^1}$. Similarly, we can obtain $\|u(0)\|_{H^1} = \|u(t)\|_{H^1}$ for $t \in [T', 2T']$, etc. until the whole interval $[0, T]$ is exhausted.

The proof of Theorem 1.2 is complete.

## 5. Proof of Theorem 1.3

In this section, motivated by the idea of [3], we will use the solitary wave solution $u_c(x, t) = \sqrt{c}e^{-|x-x_0-ct|}$, where $c > 0$ and $x_0$ is a constant, to prove that the local well-posedness of the Novikov equation fails in $B^{3/2}_{2,\infty}$.

Now we are in a position to prove Theorem 1.3.

**Proof of Theorem 1.3.** For $c > 0$, we define $u_c(x, t) = \sqrt{c}e^{-|x-x_0-ct|}$, where $x_0$ is a constant. Thus $u_c(x, t)$ is the solitary wave solution for (1.1). By using the definition of Fourier transformation, we have

$$\begin{aligned}
\mathcal{F}_x u(\xi, t) &= \sqrt{c} \int_R e^{-ix\xi} e^{-|x-x_0-ct|}\,dx \\
&= \sqrt{c}e^{-ict\xi - ix_0\xi}\left(\int_{-\infty}^{0} e^{-ix\xi} e^{x}\,dx + \int_0^{\infty} e^{-ix\xi} e^{-x}\,dx\right) d\xi \\
&= \sqrt{c}e^{-ict\xi - ix_0\xi}\left(\frac{1}{1 - i\xi} + \frac{1}{1 + i\xi}\right) d\xi \\
&= 2\sqrt{c}e^{-ict\xi - ix_0\xi}\frac{1}{1 + \xi^2}.
\end{aligned} \tag{5.1}$$

Let $c_j$ $(j=1,2)$ be constants which will be specified later. We have

$$
\begin{aligned}
\left\|u_{c_2}(0)-u_{c_1}(0)\right\|^2_{B^{3/2}_{2,\infty}} &= 8(\sqrt{c_2}-\sqrt{c_1})^2 \max\left(\int_0^1 \frac{d\xi}{\sqrt{1+\xi^2}}, \sup_{q\in N}\int_{2^q}^{2^{q+1}} \frac{d\xi}{\sqrt{1+\xi^2}}\right) \\
&= 8(\sqrt{c_2}-\sqrt{c_1})^2 \max\left(\log(1+\sqrt{2}), \sup_{q\in N}\log\frac{2^{q+1}+\sqrt{2^{2q+2}+1}}{2^q+\sqrt{2^{2q}+1}}\right) \\
&= 8(\sqrt{c_2}-\sqrt{c_1})^2 \log(1+\sqrt{2}). \qquad (5.2)
\end{aligned}
$$

From the definition of $u_c(x,t)$, we have that $u_c(x,t)=0$ if $c=0$. From (5.2), we have

$$
\left\|u_c(x,0)\right\|_{B^{3/2}_{2,\infty}} = 8c\log(1+\sqrt{2}).
$$

In fact, combining (5.1) with the definition of the Besov space $B^s_{p,r}$, we have

$$
\left\|u_c(x,t)\right\|_{B^{3/2}_{2,\infty}} = 8c\log(1+\sqrt{2}).
$$

Let $d=c_2-c_1$ and $d_1=\sqrt{c_2}-\sqrt{c_1}$, pick $c_1=1$, $c_2=1+2^{-q}T^{-1}\pi$ and $2^qTd=\pi$, then we have

$$
\begin{aligned}
&\left\|u_{c_2}(x,t)-u_{c_1}(x,t)\right\|^2_{B^{3/2}_{2,\infty}} \\
&= 8\max\left(\int_0^1 \frac{d_1^2+2\sqrt{c_1c_2}(1-\cos dt\xi)}{\sqrt{1+\xi^2}}\,d\xi, \sup_{q\in N}\int_{2^q}^{2^{q+1}} \frac{d_1^2+2\sqrt{c_1c_2}(1-\cos dt\xi)}{\sqrt{1+\xi^2}}\,d\xi\right) \\
&\geqslant 16\sqrt{c_1c_2}\int_{2^q}^{2^{q+1}} \frac{1-\cos dt\xi}{\sqrt{1+\xi^2}}\,d\xi \\
&\geqslant 16\sqrt{c_1c_2}\int_{2^q}^{\frac{3}{2}2^q} \frac{1-\cos dt\xi}{\sqrt{1+\xi^2}}\,d\xi \geqslant 1,
\end{aligned}
$$

while

$$
\begin{aligned}
\left\|u_{c_2}(x,0)-u_{c_1}(x,0)\right\|_{B^{3/2}_{2,\infty}} &\leqslant \sqrt{8\frac{(c_2-c_1)^2}{(\sqrt{c_2}+\sqrt{c_1})^2}\log(1+\sqrt{2})} \\
&\leqslant \frac{2\pi}{2^qT}\sqrt{2\log(1+\sqrt{2})} \longrightarrow 0, \quad q\longrightarrow\infty.
\end{aligned}
$$

Now we define $u(x,t)=u_{c_1}(x,t)$ and $v(x,t)=u_{c_2}(x,t)$.

We complete the proof of Theorem 1.3. □

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